Exercise 48

The Power Rule can be proved using implicit differentiation for the case where n is a rational number, $n = p/q$, and $y = f(x) = x^n$ is assumed beforehand to be a differentiable function. If $y = x^{p/q}$, then $y^q = x^p$. Use implicit differentiation to show

$$y' = \frac{p}{q}x^{(p/q)-1}$$

Solution

Suppose that the Power Rule only holds for integers and that

$$y = x^n$$

is a differentiable function, where n is a ratio of two nonzero irreducible integers.

$$y = x^{p/q}$$

Raise both sides to the power of q.

$$(y)^q = (x^{p/q})^q$$

$$y^q = x^p$$

Differentiate both sides with respect to x.

$$\frac{d}{dx}(y^q) = \frac{d}{dx}(x^p)$$

Use the chain rule to differentiate $y = y(x)$.

$$qy^{q-1}\frac{dy}{dx} = px^{p-1}$$

Solve for dy/dx.

$$\frac{dy}{dx} = \frac{p}{q}y^{-(q-1)}x^{p-1}$$

Substitute the formula for y.

$$\begin{aligned}\frac{dy}{dx} &= \frac{p}{q}(x^{p/q})^{-(q-1)}x^{p-1} \\ &= \frac{p}{q}x^{-\frac{p}{q}(q-1)}x^{p-1} \\ &= \frac{p}{q}x^{-p+(p/q)}x^{p-1} \\ &= \frac{p}{q}x^{-p+(p/q)+p-1} \\ &= \frac{p}{q}x^{(p/q)-1}\end{aligned}$$